Closing *Thurs*: HW_6A,6B (7.4, 7.5)
Closing next *Wed*: HW_7A,7B (7.5, 7.7, 7.8)
Note: Friday is a university holiday (no class)
Exam 2 is next **Thursday** (Nov. 16th)
Covers 6.4, 6.5, 7.1-7.5, 7.7, 7.8

We have learned how to integral some important situations. But despite our best efforts in 7.1-7.5**,** many, many, many integrals CANNOT be done with any of our methods. So, in many applications, we have to approximate!

7.7 Approximating Integrals

To approximate $\int_a^b f(x)dx$

1. Compute $\Delta x = \frac{b-a}{n}$.
   Label the tick marks: $x_i = a + i\Delta x$
2. Use an approximation method:

$$L_n = \Delta x[f(x_0) + f(x_1) + \cdots + f(x_{n-1})] \quad \text{(Left endpoint)}$$
$$R_n = \Delta x[f(x_1) + f(x_2) + \cdots + f(x_n)] \quad \text{(Right endpoint)}$$
$$M_n = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)] \quad \text{(Midpoint)}$$

New - Trapezoid Rule: (all the "middle terms" are multiplied by 2)

$$T_n = \frac{1}{2}\Delta x[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

New - Simpson's Rule: *n* must be even! (Alternating multiplying middle terms by 4 and 2)

$$S_n = \frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Example: (None of our methods can integrate this)

Estimate $\int_0^3 \sqrt{100 - x^3}\,dx$, using *n = 3* subdivisions.

- *Step 1*: $\Delta x = \frac{3-0}{3} = 1.\ x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$

  Step 2: Here is each method:

$$\int_0^3 \sqrt{100 - x^3}\,dx \approx L_3 = (1)\left[\sqrt{100-(0)^3} + \sqrt{100-(1)^3} + \sqrt{100-(2)^3}\right] \approx 29.5415$$

$$\int_0^3 \sqrt{100 - x^3}\,dx \approx R_3 = (1)\left[\sqrt{100-(1)^3} + \sqrt{100-(2)^3} + \sqrt{100-(3)^3}\right] \approx 28.0855$$

$$\int_0^3 \sqrt{100 - x^3}\,dx \approx M_3 = (1)\left[\sqrt{100-(0.5)^3} + \sqrt{100-(1.5)^3} + \sqrt{100-(2.5)^3}\right] \approx 29.0091$$

NEW – Trapezoid rule with n = 3.

$$T_3 = \frac{1}{2}(1)\left[\sqrt{100-(0)^3} + 2\sqrt{100-(1)^3} + 2\sqrt{100-(2)^3} + \sqrt{100-(3)^3}\right] \approx 28.8135$$

NEW – Simpson's rule with n = 6 (*n* must be even), $\Delta x = \frac{3-0}{6} = \frac{1}{2}.\ x_0 = 0, x_1 = \frac{1}{2}, \ldots$

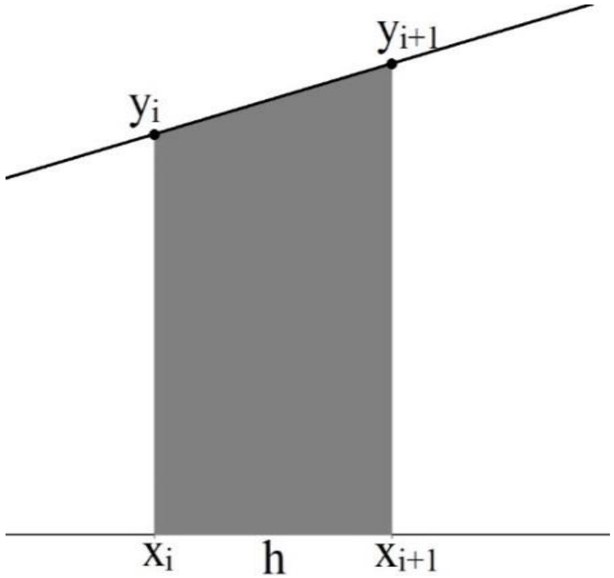
$$S_6 = \frac{1}{3}\cdot\frac{1}{2}\left[\sqrt{100-(0)^3} + 4\sqrt{100-(0.5)^3} + 2\sqrt{100-(1)^3} + 4\sqrt{100-(1.5)^3}\right.$$
$$\left. + 2\sqrt{100-(2)^3} + 4\sqrt{100-(2.5)^3} + \sqrt{100-(3)^3}\right] \approx 28.9441$$

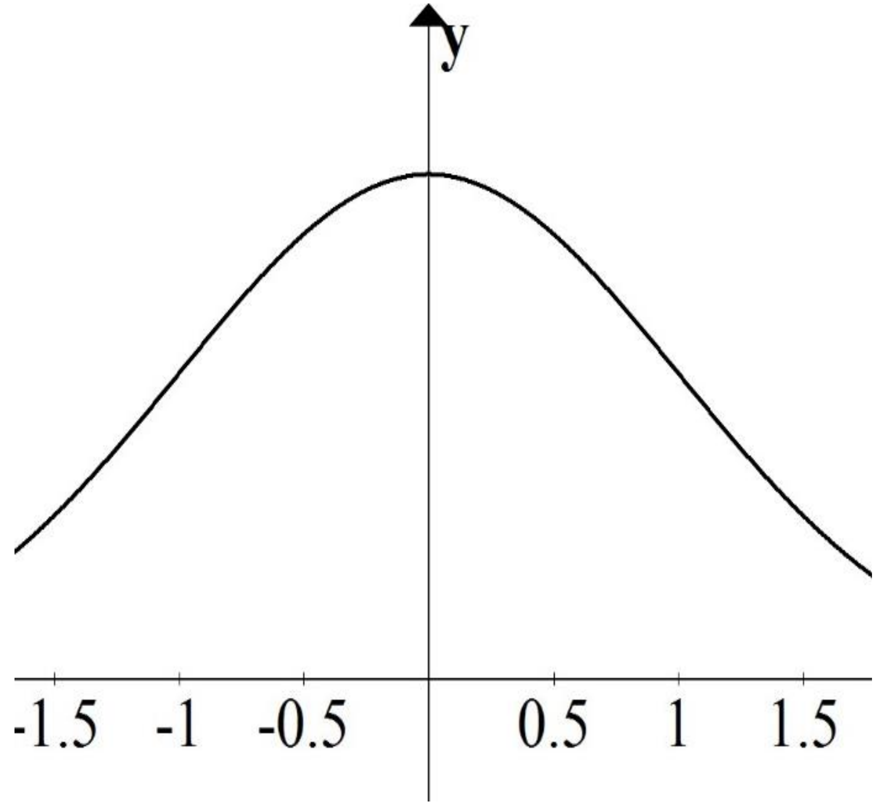
"Actual" Value (to 8 places after the decimal): 28.94418784

7.7 Derivation Notes

Trapezoid Rule:

$$\text{Shaded Area } = \frac{h}{2}(y_i + y_{i+1})$$



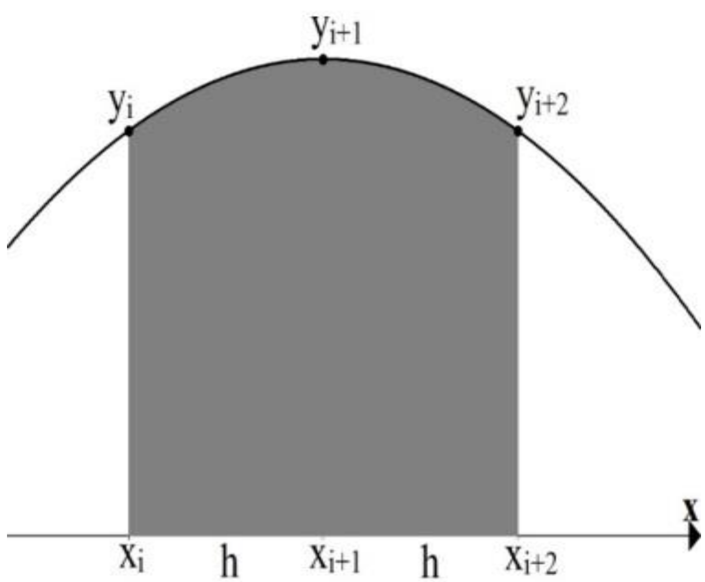


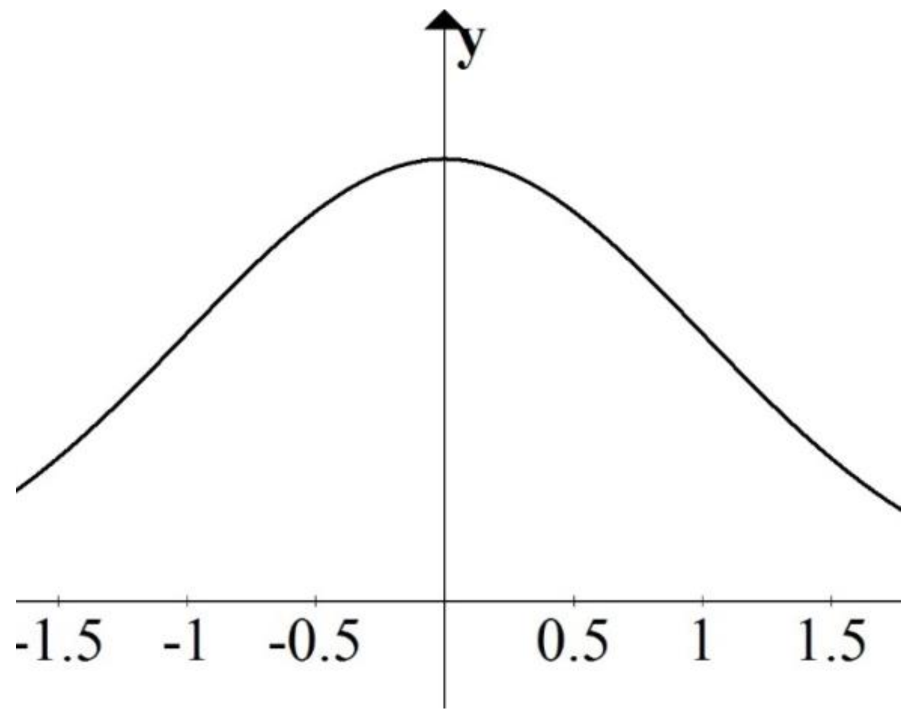
Simpson's Rule:

If the curve below is a ***parabola***,

$y = ax^2 + bx + c$, that goes through the three indicated points, then

$$\text{Shaded Area} = \frac{h}{3}(y_i + 4y_{i+1} + y_{i+2})$$





Example:
With n = 4, use both new methods to approximate (just set up)

$$\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-\frac{1}{2}x^2}\,dx$$

$\boldsymbol{\Delta x =}$ $\qquad$ $\boldsymbol{, x_0 =}$ $\qquad$ $\boldsymbol{, x_1 =}$ $\qquad$ $,$

$\boldsymbol{x_2 =}$ $\qquad$ $\boldsymbol{, x_3 =}$ $\qquad$ $\boldsymbol{, x_4 =}$

$$\frac{1}{2}\Delta x[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

$$=$$

$$\frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

$$=$$

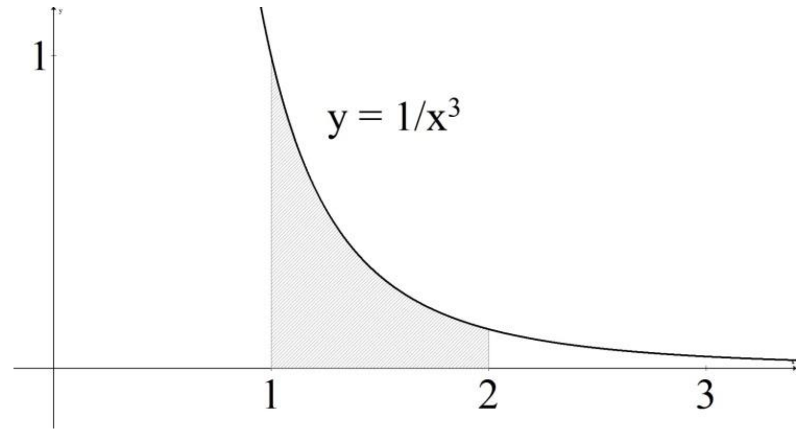
7.8 Improper Integrals *(Preview)*

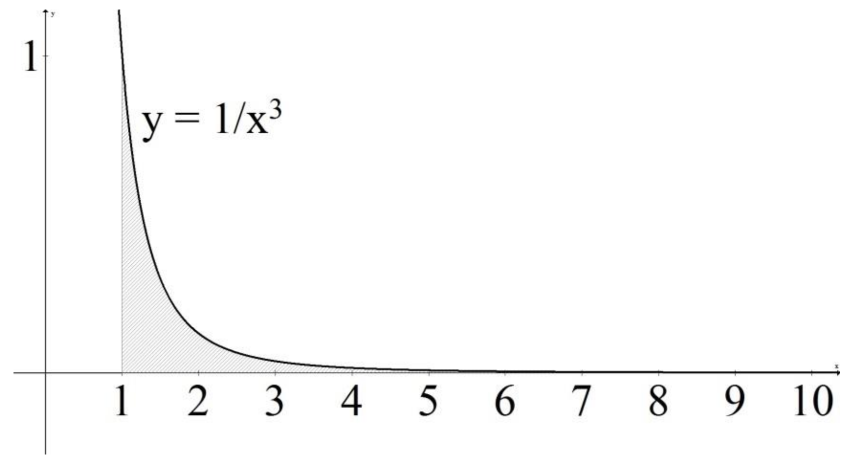
Motivation:

Consider the function $f(x) = \frac{1}{x^3}$.

Compute the area under the function…

1. …from $x = 1$ to $x = t$
2. …from $x = 1$ to $x = 10$
3. …from $x = 1$ to $x = 100$





Def'n: *Improper type 1 - infinite integral of integration*

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

If the limit exists and is finite, then we say the integral *converges*.
Otherwise, we say it *diverges*.

Example:

$$\int_1^\infty \frac{1}{x^3}\,dx =$$

Example:

$$\int_{-1}^{\infty} e^{-2x} dx =$$

Example:

$$\int_{1}^{\infty} \frac{1}{x} dx =$$

Def'n:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to-\infty} \int_{r}^{0} f(x)dx + \lim_{t\to\infty} \int_{0}^{t} f(x)dx$$

In this case, we say it *converges* only if both limits separately exist and are finite.

Example:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

Def'n: *Improper type 2 - infinite discontinuity*

If $f(x)$ has a discontinuity at x = a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$$

If $f(x)$ has a discontinuity at x = b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

If the limit exists and is finite, then we say the integral *converges*.

Otherwise, we say it *diverges*.

Example:

$$\int_0^1 \frac{1}{\sqrt{x}}dx =$$

Example:

$$\int_{0}^{2} \frac{x}{x-2}\,dx =$$

If $f(x)$ has a discontinuity at $x = c$ which is **between** a and b, then

$$\int_a^b f(x)dx = \lim_{r \to c^-} \int_a^r f(x)dx + \lim_{t \to c^+} \int_t^b f(x)dx$$

In this case, we say it *converges* only if both limits separately exist and are finite.

Example:

$$\int_0^\pi \frac{1}{\cos^2(x)} dx =$$

Limits Refresher

1. If stuck, plug in values "near" t.
2. Know your basic functions/values:

$$\lim_{t\to\infty} \frac{1}{t^a} = 0, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} \frac{1}{e^{at}} = 0, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} t^a = \infty, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} \ln(t) = \infty.$$

$$\lim_{t\to 0^+} \ln(t) = -\infty.$$

3. For indeterminant forms, use algebra and/or L'Hopital's rule

Examples:

$$\lim_{t\to 1} \frac{t^2 + 2t - 3}{t - 1} =$$

$$\lim_{t\to\infty} \frac{\ln(t)}{t} =$$

$$\lim_{t\to\infty} t^2 e^{-3t} =$$

Aside:

A few general notes on **comparison**:

Suppose you have two functions f(x) and g(x) such that 0 ≤ g(x) ≤ f(x) for all values.

(a) If $\int_1^\infty f(x)dx$ converges,
then $\int_1^\infty g(x)dx$ converges.

(b) If $\int_1^\infty g(x)dx$ diverges,
then $\int_1^\infty f(x)dx$ diverges.

You can verify that

$$\int_1^\infty \frac{1}{x^p}dx, \qquad \text{converges for } p > 1.$$

$$\int_1^\infty e^{px}\,dx, \qquad \text{converges for } p < 0.$$

And you can compare off of these to sometimes quickly tell is something is converging or diverging (without calculating anything)